Boundary Layers

We're interested here mainly in boundary layers relevant to planets, i.e. those of planetary atmospheres, oceans and fluid cores. Of these, the atmospheric boundary layer is by far the best studied, and therefore we'll rely mainly on that literature. However, the concepts will be kept as general as possible in order to be applicable to other important cases as well. There is a another class of important boundary layer problems involving aerodynamics of objects moving in fluids, i.e. boats, airplanes, birds, and the most important of all, golf balls, that we will not address here. There, the problem is mainly one of determining the drag on the object. There are also engineering applications where the boundary layer of a fluid flowing past a solid object (say, in a heat exchanger) must be understood. Since this is class is aimed at geoscientists, we ignore this class of problems here. For us, fluid boundary layers are important because they are the interfacial region between the bulk fluid and the surface, thus mediating all interactions between the two.

If we look back at the Navier-Stokes equations, we recall that the diffusivity term has the form $\frac{1}{Re}\nabla^2 u$. We said that in general this term can be neglected relative to other terms such as advection because most geophysical flows have high Re and are thus turbulent. However, there must exist some sufficiently small length scale such that Re is sufficiently small such that the diffusion term becomes important. This region is the boundary layer, and is the thin region in the vicinity of a solid (or liquid in the case of gas flows) where diffusion is non-negligible. It is characterized by strong gradients in velocity due to the no-slip requirement (see definition below).

Boundary layers can be either laminar or turbulent. Strictly speaking, even a turbulent boundary layer will have some finite region adjacent to the surface where the fluid flow is laminar. The *free stream* is the region outside the boundary layer; exactly where one ends and the other begins is somewhat arbitrary, and may not be the same depending on whether one is interested in a momentum or energy boundary layer. One common definition of the extent of the BL for momentum is the region where $u \leq 0.99\,u_\infty$, where u is the local velocity and u_∞ is the free stream velocity. You can conceive of a similar definition for, say, temperature, and you might then recognize that the dividing line for temperature can be different from that for velocity. Turbulence in the boundary layer is the result of two main processes: shear and buoyancy.

For simplicity, we'll focus on 2-D flow (see Fig. 1), which allows the equations to be easier to use while maintaining the relevant physical interpretations. One of the spatial dimensions will always be height, and the other is drawn parallel to the mean fluid flow (e.g. the mean wind).

Definitions

Boundary layer: A region of fluid in the immediate vicinity of a bounding surface where strong gradients of velocity (and, potentially, other variables such as temperature) occur. Outside the boundary layer, the fluid has little to no awareness of the bounding surface.

Surface layer: The region of the boundary with the strongest gradients in velocity (and, therefore, temperature, moisture, etc.). Typically the surface layer is ~10% of the thickness of the boundary layer. Because of the very sharp gradients in momentum, energy, moisture, etc., the surface layer is the bottleneck in the transport of these quantities between the surface and the free stream.

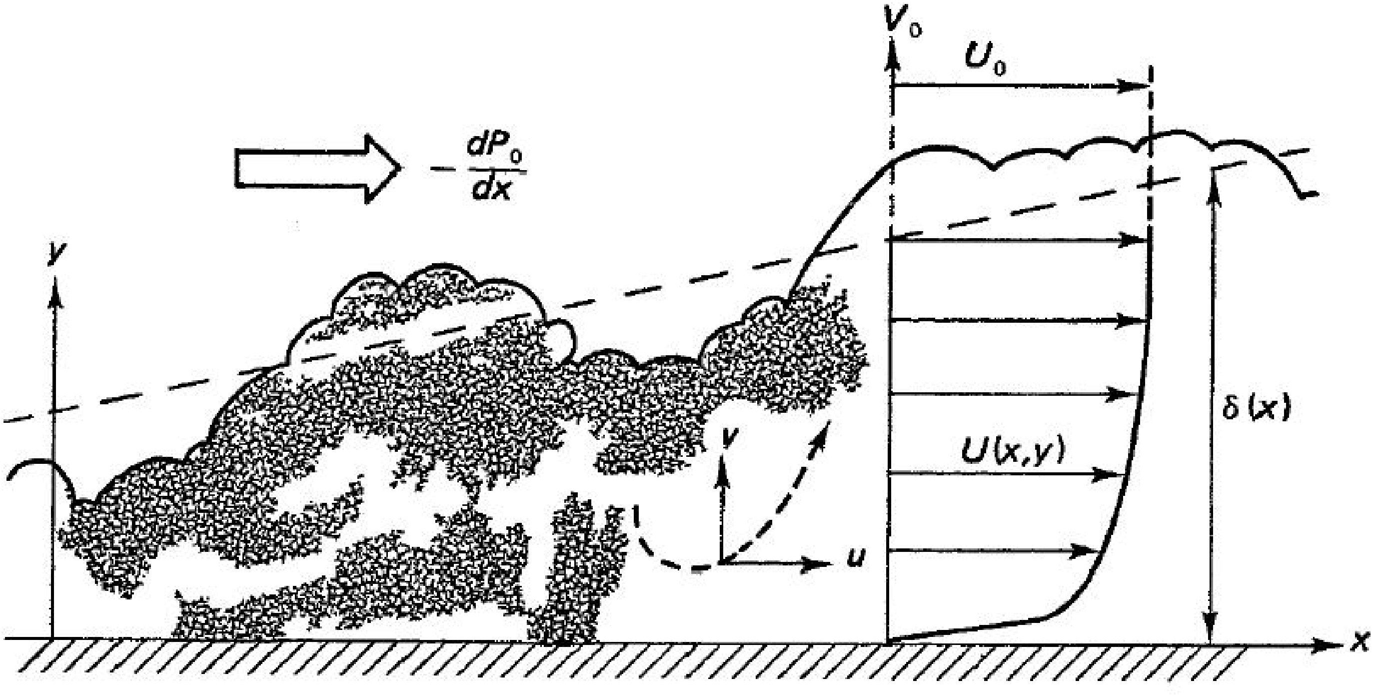

Figure 5.16. **Definition sketch of plane boundary-layer flow.**

Figure 1: Schematic of boundary layer. Notice the thin surface layer where the gradients in U are highest. The thickness of the boundary layer is δ.

No-slip condition: The requirement that the relative velocity between a fluid and solid is zero at the fluid/solid interface. This means that, for example, the wind speed is zero directly adjacent to a solid surface like a parking lot; this wind speed then strongly increases with altitude for some height above the parking lot, a region which represents the boundary layer.

Momentum: Recall that momentum p is defined as $p = mv$. Throughout these notes, however, almost all math is done on a per unit mass basis, and therefore momentum per unit mass is the same as velocity v.

Analogy to Turbulence

Note: this section closely follows *A First Course in Turbulence* by Tennekes and Lumley (1972). There is a close analogy between the *spatial* structure of a turbulent boundary layer and the the *spectral* structure of turbulence. At sufficient large Re, the structure of the turbulent boundary layer is independent of viscosity, just as for the large scale structure of turbulence. At spatial scales much smaller than the scale of the problem, the surface of the boundary layer serves as a momentum sink due to viscosity, just like eddies at the Kolmogorov scale. We define some parameters:

δ: This is the scale height of the boundary layer. Analogous to L in turbulence.

l_s: This is the scale height of the surface layer, determined by $\ell_s = {}^{\nu}\!/_{w'}$ where w' is a characteristic turbulent vertical velocity (vertical defined as perpendicular to the surface). Analogous to η in turbulence.

x: Distance along the surface.

y: Height above the surface.

u: Horizontal velocity (i.e. in x-direction or parallel to surface). One of the key results that we seek is du/dy.

w: Vertical velocity (i.e. in y-direction or perpendicular to surface). Note that we assume that $\overline{w} = 0$, but $w' \neq 0$.

For sufficiently large Re, there will exist a distance y such that $y \ll \delta$ and $y \gg \ell_s$. At this length scale y, viscosity is not sufficiently influential to affect the flow since the length scale is larger than than surface layer thickness, but δ is too large to affect the flow. Therefore, only y is the relevant length scale in the problem for these intermediate length scales, and defines the *inertial sublayer* which is analogous to the inertial subrange in turbulence. Using simple dimensional analysis, we deduce that:

$$\frac{du}{dy} \sim \frac{w'}{y}$$

which can be integrated to:

$$\frac{u}{w'} \sim \ln y$$

This logarithmic velocity profile is an important prediction in boundary layers, and turns out to work very well for shear-driven flows in a neutrally stable fluid.

Transport of Momentum, Heat...

One important goal of boundary layer theory is to determine the flux of important quantities between the surface and the free stream fluid. To estimate these values, we re-visit the turbulent diffusion equations from the turbulence lecture. We found that the turbulent transport of any quantity a (where a can be temperature, momentum, water vapor...) can be given by an equation similar to Fick's Law:

$$\overline{w'a'} = -K_a \frac{d\overline{a}}{dz} \tag{1}$$

where the eddy diffusivity K_a is given by:

$$K_a = {}^1\!/_2\, w'\, h \tag{2}$$

For the boundary layer, we use the following approximations: $d\overline{a} \sim \triangle a = a_\infty - a_{surf}$; $dz \sim \delta$; $w' \sim u' \sim {}^1\!/_{10}\overline{u}$; $h \sim {}^1\!/_{10}\delta$, where a_∞ is the value of a in the free stream, and a_{surf} is the value of

a at the surface. We have also used the rule of thumb that the turbulent intensity $u'$is 10% of the mean wind speed $\overline{u}$. If we substitute these OOM estimations into Eqs. (1) and (2), we find:

$$\overline{w'a'} = \frac{1}{200}\,\overline{u}\,(a_\infty - a_{surf}) = 0.005\,\overline{u}\,(a_\infty - a_{surf}) \tag{3}$$

This result is in very good agreement with empirically-derived relations. Very smooth surfaces (e.g. ocean, sea ice) have less turbulence than assumed here, and have transport rates about a factor of 3 lower. Very rough terrain, with a mixture of fields, houses, clumps of trees, etc., can have transport rates about a factor of 3 to 5 higher than this because of greater turbulence than assumed here. These estimates are better for shear-driven turbulence; when buoyancy and hence convection becomes important, transport rates can be much higher. Also note that this analysis assumes that the process that limits transport is turbulent transport from the surface layer to the free stream. Can you show that diffusion across the surface layer is much faster than this turbulent transport?

Examples: air mass formation; loss of seasonal ice cap on Mars; limit on CO2 transport to deep ocean (assume a two-layer stratified ocean).

Bowen Ratio

The Bowen Ratio is the ratio between sensible and latent heat fluxes (in units of W m^{-2}) from a surface, i.e.:

$$B = \frac{Q_s}{Q_l} \tag{4}$$

where Q_s is the sensible heat flux, and Q_l is the latent heat flux. For Earth's atmosphere, the globally averaged exchange of energy between the boundary layer and surface is:

$$B = \frac{\overline{w'T'}\,\rho c_p}{\overline{w'q'_v}\,L} \tag{5}$$

where ρc_p is that of air, L is latent heat of evaporation of water, and q_v is the water vapor mass concentration. We can estimate the value for Earth by using Eq. (3) in Eq. (5), which yields:

$$B = \frac{\overline{w'T'}\,\rho c_p}{\overline{w'q'_v}\,L} = \frac{0.005\,\overline{u}\,(T_\infty - T_{surf})\,\rho c_p}{0.005\,\overline{u}\,(q_{v,\infty} - q_{v,surf})\,L} = \frac{(T_\infty - T_{surf})\,\rho c_p}{(q_{v,\infty} - q_v^*(T_{surf}))\,L} \tag{6}$$

where q_v^* is the saturated water vapor concentration, i.e. we're assuming that the layer of air adjacent to the surface is saturated at the surface temperature. This is a very good approximation over the ocean, not a bad approximation for fairly wet terrain, and a horrible one for arid soils (and we've discounted evapotranspiration from plants, which can be very important). Remembering that our planet is mostly ocean, and after that, mostly tropical, we'll assume this is OK for now. Note another feature of this estimate: the variable coefficient that we choose to be 1/200 disappears from our estimate because we're estimating ratios. In many cases, OOM can yield the *relative strengths* of processes more easily than *absolute values* because some variable or difficult-to-estimate quantities will cancel.

So let's take a stab at some numbers OOM: $T_\infty - T_{surf}$ we can get some idea from thinking about walking over a surface in bare feet, compared to air temperature. Obviously there's a lot of variability depending on the type of surface, but it's probably more than 5 K and less than 20 K, so I'll go right in between (geometrically) and use 10 K; for absolute numbers (which we need next), we'll choose 30 C and 20 C. If we assume that the air has a relative humidity of 70%, this yields a value of 0.02 kg m^{-3} for $q_{v,\infty} - q_{v,surf}$ (can you estimate this?). Substituting leads to the estimate:

$$B = {}^{1}\!/_{4}$$

which is remarkably accurate (I didn't cheat by working backwards - really!) - the globally averaged value is just about ${}^{1}\!/_{3}$ which is exactly the same number OOM!

Examples: Lakes on Titan; difference in Bowen ratio over land and water