

**ES265 Order of Magnitude Phys & Chem**  
**Lecture Outline 6**  
**Convection**

Convection deals with moving fluids in which there are spatial variations in temperature or chemical concentration. In *forced convection*, these variations do not influence the flow significantly, and heat/chemistry are advected around passively. By contrast, in *free convection*, it is the variations in heat/chemistry which control the fluid motion. We will focus on the latter situation.

Part of the reason free convection is complicated is that there are two governing equations (plus mass conservation) which have to be satisfied simultaneously.

Conservation of energy gives (see Week 2)

$$\frac{\partial T}{\partial t} + u\nabla T = \kappa\nabla^2 T + \frac{H}{\rho C_p} \quad (1)$$

An equivalent equation can be written if it is some chemical species, rather than heat, which is being conserved.

Lateral variations in temperature give rise to *buoyancy forces* which drive convection. A parcel of material which is  $\Delta T$  hotter than the surroundings will have a density contrast of  $\rho\alpha\Delta T$  where  $\rho$  is the background density and  $\alpha$  is the thermal expansivity. It will experience an upwards force per unit mass of  $g\alpha\Delta T$ . *Compositional convection* can also be modelled, except here the density contrast is given by  $\rho\alpha_c c$ , where  $c$  is a concentration variable and  $\alpha_c$  is the compositional expansivity. In the *Boussinesq* approximation, variations in density except those due to thermal/compositional effects are ignored. For compressible materials (e.g. gases) this is a bad idea!

Conservation of momentum therefore gives (see Week 3)

$$\frac{\partial u}{\partial t} + u\nabla u = \nu\nabla^2 u - \frac{\nabla P}{\rho} + g\alpha\Delta T \quad (2)$$

Note that equations (1) and (2) are coupled because  $u$  depends on  $\Delta T$  and  $T$  depends on  $u$ .

As usual, a good way to start is by balancing terms. Let's look for some characteristic timescales.

For heat, the advection timescale is  $d/u$  and the diffusion timescale is  $d^2/\kappa$ , so the ratio is  $\sim ud/\kappa$ . If this number  $\gg 1$ , then advection dominates heat diffusion (recall that  $u$  will approach zero at boundary layers, so diffusion cannot be entirely ignored!).

For momentum, the equivalent ratio is  $\sim ud/\nu = Re$ , the Reynolds number. This tells us whether flow is laminar or turbulent.

If the Reynolds number is small (laminar regime), then we can work out whether heat or momentum diffuses faster by looking at the ratio of the two diffusion timescales. This ratio is called the *Prandtl number*:

$$Pr = \frac{\nu}{\kappa}$$

The Prandtl number is a property of the fluid, and for many gases and liquids it is  $\sim 1 - 10$  (i.e. momentum diffuses more rapidly than heat). Liquid metals have  $Pr < 1$ . For the Earth's mantle, the Prandtl number is effectively infinite. If  $Pr = 1$  then the thermal and mechanical boundary layer thicknesses are equal.

Continuing at low  $Re$ , in steady flow we can balance equation (2) and obtain a characteristic velocity

$$u \sim \frac{g\alpha\Delta T d^2}{\nu}$$

where  $d$  and  $\Delta T$  are a characteristic lengthscale (e.g. layer thickness) and temperature contrast. This in turn gives us an advective timescale  $t \sim d/u \sim \nu/g\alpha\Delta T d$ .

For (free) convection to occur, the advective timescale has to be shorter than the heat diffusion timescale  $d^2/\kappa$  - otherwise a rising blob would lose all its heat. This allows us to define a characteristic number describing the vigour of convection, the *Rayleigh number*:

$$Ra = \frac{g\alpha\Delta T d^3}{\kappa\nu} \quad (3)$$

Another way of thinking about  $Ra$  is that it is the competition between thermal buoyancy and viscous dissipation: if the former wins, convection will occur, whereas if the latter wins, the fluid will not move and heat will be transferred by conduction.

For convection to occur, the Rayleigh number must exceed a critical value ( $Ra_{cr}$ ,  $\sim 10^3$ ). Higher Rayleigh numbers produce more vigorous convection. The Rayleigh number controls essentially all aspect of convection.

*Example:* The Earth's mantle has a viscosity of  $10^{17} \text{ m}^2 \text{ s}^{-1}$ . What is its Rayleigh number?

*Example:* If we wanted to simulate the Earth's mantle using honey, how big a pot of honey would we need?

*Example:* What is the Rayleigh number of a magma lake?

Different boundary conditions require slightly different definitions of  $Ra$ . Equation (3) applies to a situation with a fixed temperature difference across the layer. If the heat flux  $F$  is specified, you exchange  $\Delta T$  for  $Fd/k$  where  $k$  is the thermal conductivity. If the heat production rate  $H$  is specified, you exchange  $\Delta T$  for  $Hd^2/k$ , and so on.

*Example:* If all planetary mantles have the same viscosity as the Earth's, how small is the smallest planet which will undergo convection?

A significant advantage of  $Pr \sim 1$  fluids is that you can focus on the momentum equation (2), because heat just advects and diffuses at the same rate as momentum.

As an example, consider the steady rising of fluid above a localized heat source to form a 2D plume. The fluid has  $Pr \sim 1$  and we'll assume it's at some intermediate  $Re$ .

Neglecting pressure variations and assuming steady flow, there are three relevant terms in equation (2) which can be balanced to give us

$$u \sim \frac{\nu z}{\delta^2} \quad , \quad u \sim \frac{g\alpha\Delta T \delta^2}{\nu}$$

(Where did that  $z$  come from?) Unfortunately, there are three quantities that vary with  $z$ :  $u$ ,  $\delta$  and  $\Delta T$ . So we need a third equation, which we get by asserting that the rate of heat transport is independent of  $z$  (as it must be in steady state). This gives us, for a 2D flow:

$$\rho C_p u \delta \Delta T \sim F$$

where  $F$  is a constant ( $\text{W m}^{-1}$ ). Putting these three equations together, we get

$$\delta \sim \left( \frac{\nu^3 \rho C_p}{g \alpha F} \right)^{1/5} z^{2/5}$$

and

$$u \sim \left( \frac{g \alpha F}{\rho C_p} \right)^{2/5} \frac{z^{1/5}}{\nu^{1/5}}$$

*Example* A hydrothermal vent is generating heat at a rate of 1 MW/m. What are the temperature anomaly, width and velocity 1 km above the plume?

### Boundary layer theory

A vigorously convecting fluid will consist of three regions: a wide zone in the centre, in which convection is vigorous, heat is transferred by advection and the temperature is close to constant (adiabatic); and two thin *boundary layers* at the top and bottom, where velocities are small and heat is transferred by conduction. The thickness of the boundary layers,  $\delta$ , obviously plays an important role in determining how much heat is transferred. For instance, on Earth we can assume that  $\delta$  is given by the thickness of the lithospheric plates.

The nature of these boundary layers will depend on both  $Re$  and  $Pr$ . If  $Re \gg 1$ , then the standard turbulent boundary layer will develop. In this case, heat transfer will occur by conduction across the thin sub-layer. Using the arguments from Week 4, the turbulent heat transport will equal the diffusive heat transport, so we can write

$$\rho C_p \overline{w'T'} \sim \rho C_p w' \delta \frac{d\bar{T}}{dz} \sim k \frac{dT}{dz}$$

where the boundary layer thickness is  $\delta$ . This in turn gives

$$\delta \sim \frac{\kappa}{w'}$$

Thus, if we know the thermal diffusivity and the velocity fluctuations, we can infer the turbulent boundary layer thickness. Clearly, these layers are going to be very thin unless  $w'$  is small.

For the case when  $Re \ll 1$ , the thickness of the boundary layers is similarly determined by the balance between advection and conduction of heat, except here the flow is laminar. There are two ways of determining the boundary layer thickness  $\delta$  in this case.

One is to argue that the boundary layer must be just at the critical Rayleigh number (if it were any thicker, it would be participating in convection). Since the interior is isothermal, the temperature contrast across the boundary layer is  $\sim \Delta T$  and we can write

$$\frac{g \alpha \Delta T \delta^3}{\kappa \nu} \sim Ra_{cr}$$

which gives

$$\delta \sim d Ra^{-1/3} \tag{4}$$

The alternative is to assert that the behaviour of the boundary layer doesn't care about how deep the convecting fluid beneath it extends. In this case,  $\delta$  must be independent of  $d$ , which

means that equation (4) has to be correct. This approach makes the extension to other boundary conditions more obvious.

*Example:* Use the thickness of the lithosphere to estimate  $Ra$  for the Earth's mantle.

*Example:* How does  $\delta$  vary with  $Ra$  for convection when the heat flux  $F$  is specified?

The scale of convective motions in the fluid is determined by  $\delta$  e.g. upwelling blobs tend to be roughly  $\delta$  wide. So we can use  $\delta$  to determine characteristic timescales and velocities.

The characteristic cooling timescale of a blob is just  $\delta^2/\kappa$ . If we balance this against its transit timescale  $d/u$ , we get  $u \sim d\kappa/\delta^2$  which allows us to write

$$u \sim \frac{\kappa}{d^2} Ra^{2/3} \quad (5)$$

*Example:* What are characteristic convective velocities in the Earth's core? What might be wrong with our assumptions?

So far we have assumed that the viscosity is constant, while for real fluids it is often temperature-dependent. The result is a thicker boundary layer at the top, because the fluid is cold and viscous, and a thinner boundary layer at the bottom. Equation (4) still applies, there is just an extra multiplicative constant which depends on how strongly temperature-dependent the fluid is.

Convection in porous media is governed by a similar dimensionless parameter to (3), except that rather than  $d^3$  we use  $dk_{perm}$  where  $k_{perm}$  is the permeability in  $m^2$  (think of it as the effective cross-sectional area available for fluid flow). Since typical terrestrial permeabilities are in the range  $10^{-8} - 10^{-16} m^2$ , it is apparent that convection in porous media requires either very low fluid viscosities or very large buoyancy contrasts.