Fluid Mechanics

Relevant Dimensionless Quantities

Two important dimensionless quantities related to fluid flow:

$$\text{Reynolds number Re} = \frac{\rho u l}{\mu} = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\text{kinetic energy}}{\text{mechanical energy loss}} \tag{1}$$

$$\text{Froude number Fr} = \frac{u}{\sqrt{gl}} = \frac{\text{inertial force}}{\text{gravitational force}} = \frac{\text{kinetic energy}}{\text{gravitational potential energy}} \tag{2}$$

Re is relevant to almost all flows that we'll discuss here, while Fr is relevant primarily to open channel flows such as rivers.

These dimensionless quantities are important because flows with the same Re will behave similarly (apart from a scaling factor). Thus, experiments done on flow of water in a tube can be applied to flows of air in a wind tunnel even though ρ is very different.

General (Mechanical) Energy Equation

Define an *incompressible* flow: a flow in which the fluid can be assumed to have a constant density (this is not strictly speaking correct, but is good enough for most people, including us).

Define a *streamline*: a line that is everywhere tangent to the velocity vector in a flow.

Define a *steady state* flow: a flow where the fluid properties (e.g. velocity, pressure) at any given point in space (x,y,z) are constant with time. Put another way, pressure $p = p(x, y, z)$, with no dependence on t.

At steady state, a streamline traces the path of a fluid element. By writing down a mechanical energy conservation equation for this fluid element, we find that:

$$\frac{\Delta p}{\rho} + \frac{\Delta\left(u^2\right)}{2} + g\Delta z = w - gh_L \tag{3}$$

where: p is pressure; ρ is density; u is fluid speed; z is height; w is the work done *on* the fluid element; and h_L is the head loss (i.e. the frictional losses caused by viscous and turbulent dissipation of energy), which we'll discuss in more detail later.

On the LHS of Eq. (3), the first term is the energy associated with the pressure of the fluid element; the second term is the kinetic energy; the third term is the gravitational potential energy. On the RHS of Eq. (3), the first term is the work done on the fluid, say by a pump. A negative w can occur if work is *extracted from* the fluid, e.g. used to drive a machine, such as a windmill. The second term is the loss of mechanical energy from the fluid element by the conversion of the flow's kinetic energy into heat energy of the molecules in the flow. All Eq. (3) really says is that the change in mechanical energy of a fluid element between two points (i.e. the LHS) must be related to work done on the fluid and any losses that occur. If w and gh_L are assumed to be zero, the equation simplifies to the well-known Bernoulli Equation:

$$\frac{\Delta p}{\rho} + \frac{\Delta\left(u^2\right)}{2} + g\Delta z = 0 \tag{4}$$

The Bernoulli Equation also highlights another aspect of such flows, and that is that one can trade mechanical energy among the three forms (pressure, kinetic and gravitational potential), i.e. a loss of gravitational potential energy (fluid falls to lower elevation) is balanced by an increase in either pressure or kinetic energy (or both). It is important to note the applicability of Eq. (3):

- Applies only along one streamline (i.e. the equation follows one fluid element from one location to another; you can't change the fluid element of interest)
- Applies only to a steady state flow
- Applies only to incompressible flow. Most liquids can be considered incompressible for reasonable ranges of pressure. Gases are obviously compressible *fluids*, but their *flow* can be considered incompressible provided that pressure and temperature changes are small and velocities are low, generally $M \lesssim 0.2$ where M is the Mach number.

Because the fluid is assumed to be incompressible, this causes a de-coupling of the mechanical energy equation from the thermal energy. In practical terms, this means that changes in the temperature of the fluild has no bearing on its mechanical energy, and hence must be solved using a separate equation. This occurs in incompressible flow because there is no way that thermal energy can generate mechanical energy. This is only possible in compressible flow, where, e.g., an expanding gas can generate high pressure which then pushes on a piston, as in an internal combustion engine.

Examples: high water mark for a dam release; volcano erupting; Pitot tube for measuring gas flows; the incorrect explanation of aircraft lift.

Internal flows

So far the discussion has been very general. We now focus on internal flows, where the fluid is *completely surrounded* by a solid, e.g. flows of liquids and gases through pipes and ducts, for which we can quite easily estimate the frictional losses. We will discuss channel flows later, where there is a free surface, such as in a river. Internal flows are classified as *laminar* (for $\mathrm{Re} < 2300$) or *turbulent* ($\mathrm{Re} > 4000$). For Re in between these values, the flow has a tendency to switch unpredictably between the two regimes; this is termed the *transition regime.* Friction is the conversion of kinetic energy to heat energy, and so it seems reasonable to write:

$$gh_L = K\,u^2 = f\,\frac{L}{D}\,u^2 \tag{5}$$

where K is a proportionality constant describing the conversion of kinetic energy (u^2) to heat. K is typically written as a product of two numbers, f, the friction factor, and L/D, the ratio of the pipe length L to its diameter D. Longer pipes, and pipes with smaller diameters, exhibit larger frictional losses, which is intuitively sensible. The friction factor f depends in detail on things like the smoothness of the pipe and Re, but across many orders of magnitude of both parameters, f tends to vary only by about an order of magnitude, from 0.01 to 0.2. For natural systems, where very smooth objects are rare, we can choose a value that's closer to the top of this range, so we'll choose $f = 0.1$ as a characteristic value.

For non-circular pipes, we replace the diameter D in Eq. (5) with a hydraulic equivalent diameter D_h:

$$D_h = \frac{4\,Area}{Perimeter} \tag{6}$$

This definition yields $D_h = D$ for circles, as we should expect.

When a tube bends, contracts, expands, or has an obstacle within it, there are additional frictional losses. These are typically expressed as K values (Eq. 5), and are tabulated from experimental data. These values range widely, but are typically on the order of 10^{-1} to 10^{0}.

Examples: power of a human heart (consider the diffusional length scale for oxygen to derive the length scale for the smallest blood vessels); power consumed in a slow (10 m/s) and fast (100 m/s) wind tunnel; typical velocity in a lava tube; volcanoes revisited (this time with frictional losses).

Channel Flows

If we assume that the depth of the flow and flow velocity are approximately constant, then the physics governing channel flow simplify very nicely. If this is not a good assumption, then the problem requires more detailed analysis, at which point you will end up in a morass of semi-empirical formulas.

So if we return to Eq. (3), we find that $\triangle u = 0$, $\triangle p = 0$, $w = 0$, which leaves only two terms: the gravitational potential energy and frictional losses. So under these assumptions, open channel flows convert gravity into frictional heating, which is very accurate. So now we are curious about calculating the velocity u as a function of the river's characteristics. Since it's driven by gravity, we'll assume that g and the channel slope $S = \tan\theta$ are important (note: S is dimensionless!). Perhaps a length scale l is important as well - a very tiny channel will have a large surface area (i.e. area where water contacts the side walls) compared to the volumetric flow, and thus a more drag, compared to a larger channel. So we have u, g, l and S and three dimensions. Buckingham Pi tells us that the dimensionless number looks something like:

$$\frac{u}{\sqrt{gl}} \sim c \tag{7}$$

where c is a constant. What happened to S? The problem with S is that it's dimensionless, so it can go anywhere. If we think physically about where it should go, we would probably naturally conclude that it would have to included as something like (after re-arrangement):

$$u \sim cS\sqrt{gl} \text{ or } c\sqrt{glS} \text{ or...}$$

since larger S must lead to larger u. We can't actually tell the difference using dimensional analysis. Turns out that $u \sim S^{\frac{1}{2}}$, which can be discovered by experiment or by doing a more rigorous derivation. Also, it turns out that $l = D_h$ that we defined in Eq. (6) above. Lastly, c turns out to vary between about 5 for small, rough channels and 15 for large, smooth channels. We'll choose $c = 10$ as an approximate intermediate value. With these refinements, we derive a final equation:

$$u \sim 10\sqrt{gD_hS} \tag{8}$$

Note that D_h involves the cross-sectional area A and perimeter P of the flow, which means it includes information about the depth of the flow. Therefore, one of (u, volumetric flow rate Q, and depth h) must be known to solve for the other two. Knowing the geometry of the channel will yield the second equation necessary to solve for the two unknowns.

Examples: warming of a river from friction relative to sunlight; depth of river as function of slope and width (check using data); erosive energy of rivers; how long does it take water to travel from the Sierra Nevada range to LA via the LA Aquaduct (it is mainly gravity driven); how large are storm sewers; how much water was discharged to form Martian outflow channels?

Relationship to Fr

Recall from Eq. (2) that the Froude number Fr is given by $\mathrm{Fr} = \frac{u}{\sqrt{gh}}$, which looks a lot like the result from our dimensional analysis, except I've cheated and replaced the (undefined) length scale l by the depth of the flow h, which makes it not exactly the same as what we found. It turns out that, just like Re divides different regimes (laminar and turbulent), so does Fr. $\mathrm{Fr} < 1$ is defined as subcritical, while $\mathrm{Fr} > 1$ is defined as supercritical. The speed of a surface gravity wave in water is $v = \sqrt{gh}$(see section on Waves), so supercritical flow is one where the information about a perturbation is unable to travel upstream faster than the flow sweeps it downstream, thus the flow doesn't know about an obstacle ahead of time. On the other hand, for subcritical flow, gravity waves are able to propagate upstream, so the flow has information about what is upcoming.