EART265 Order of Magnitude Phys & Chem
Lecture Outline 3
Heat Transfer and Temperature

Introduction

So far, we have not really had to memorize any equations. In studying heat transfer, it is actually helpful to learn one or two. This is because an equation acts as a **mnemonic**: it contains a lot of information, which otherwise you would have to remember piecemeal.

To deal with equations, you need to learn three important skills, examples of which you will see below:

(i) **Identifying** the processes represented
(ii) **Approximating** differential terms
(iii) **Balancing** the leading (most important) terms

Heat Production

Last week we were introduced to the concept of specific heat capacity (C_p), with units of J kg^{-1} K^{-1}. For gases $C_p \sim 3R$, where R is the gas constant (≈ 10 J mol^{-1} K^{-1}), and for solids the numbers are generally similar, though traditionally expressed in different units. For solids the product $\rho \mathrm{C_p} \approx 3$ MJ m^{-3} K^{-1} is a pretty good approximation.

The change in temperature ΔT of a substance depends on its specific heat capacity C_p and the amount of energy added/subtracted ΔW:

$$\Delta W = m C_p \Delta T \tag{1}$$

where m is the mass of the substance. If a phase change is involved, an additional energy input mL is required, where L is the latent heat (J kg^{-1}).

Example An adiabatic gas cools as it rises, because the pressure drops and it therefore expands. Use the concept that pressure=energy per unit volume to determine the *lapse rate* $\Delta T/\Delta z$ of a perfect gas.

If we consider a system in which no heat is lost, then if the rate of heat production (in W m^{-3}) is H, equation (1) may be written as

$$\rho C_p \frac{\partial T}{\partial t} = H \tag{2}$$

where ρ is the density. Higher rates of heating (RHS) result in faster temperature changes (LHS).

Example If the cooling system on a 128-core processor failed, how long would you have to get out of the room?

Heat Transfer

In reality, heat production is usually balanced by some kind of heat transfer. Heat transfer takes place when spatial variations in temperature occur, and happens via **conduction**, **radiation** or **advection**. A special form of advection is **convection**, where the fluid motion is itself driven by temperature differences.

Conduction

For any diffusive process, the flux depends on the concentration gradient and a constant of proportionality. This is Fick's law and for heat flux may be expressed as:

$$F = -k \nabla T \tag{3}$$

where F is the heat flux (W m^{-2}), k is the thermal conductivity (W m^{-1} K^{-1}) and T is the temperature. NB "flux" means per unit area (W m^{-2}); "flow" is the total power (W).

The thermal conductivity is directly related to the **thermal diffusivity** κ by $k = \kappa\rho C_p$. The thermal diffusivity for non-metals depends on the mean free path and velocity (see Week 2).

Example Estimate the thermal conductivity of metal by considering boiling a pan of water.

Combining equations (1) and (3) and considering the heat flows within, into and out of a small box, we can derive a more general equation

$$\rho C_p \frac{\partial T}{\partial t} = k\nabla^2 T + H \tag{4}$$

The first term on the RHS is the effect of conductive heat flow, the second is due to heat production.

Example Estimate H (in W kg^{-1}) for the Earth, the Sun, and a human being.

Approximating

Equation (4) gives the relationship between cooling rate, heat flow and heat production. We can **approximate** the importance of each of these terms by getting rid of all the nasty differential signs:

$$\frac{\partial T}{\partial t} \sim \frac{T}{t} \quad , \quad \nabla T \sim \frac{T}{d} \quad , \quad \nabla T^2 \sim \frac{T}{d^2} \quad , \cdots$$

where it is now understood that T,t,d, $\cdots$ represent characteristic temperature-, time- and length-scales for the particular problem that we are interested in.

Having made these simplifications, equation (4) ends up as

$$\rho C_p \frac{T}{t} \sim k\frac{T}{d^2} + H \tag{5}$$

and we can now look at situations in which one of these terms can be neglected.

If we are in *steady state*, then the LHS of equation (5) is zero and heat production is balanced by heat conduction. We obtain

$$T \sim \frac{Hd^2}{k}$$

where here T really represents the temperature *difference* driving conduction (this explains the disappearance of the minus sign!). Higher heat production and lower thermal conductivities lead to higher internal temperatures.

Example What's the internal temperature of the Earth? Do you believe the answer?

Example What's the internal temperature of an elephant? What do you conclude?

Example What's the minimum feasible size of a mammal?

If heat production is negligible, we balance heat conduction against temperature change in equation (5) and obtain

$$t \sim \frac{d^2}{\kappa} \tag{6}$$

This gives the time it takes a temperature change to propagate a given distance d in a conductive medium. An identical expression can be found by considering the heat energy contained in a box of given dimensions, divided by the characteristic heat flow.

Equation (6) is **extraordinarily important**, because it applies to any diffusive process: chemical, magnetic, viscous, you name it. If you know the diffusivity, you can derive the diffusion timescale from a known lengthscale, or vice versa.

Example How long does it take the Moon to cool?

Example How deep inside the Earth does the drop in temperature during an ice age penetrate?

Example Estimate the change in temperature around the fault surface during a magnitude-5 earthquake.

Example By considering a periodic surface heat flux F, demonstrate that the near-surface temperature change depends on a quantity $(k\rho C_p)^{1/2}$, known as the *thermal inertia*.

Example A body in a library cools to ambient in 10^5 s. What is the (pre-death) internal heating rate implied if internal body temperature is ~ 20 K above ambient?

Advection

Heat can also be transported by motion. We can add an extra term to equation (4):

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa \nabla^2 T + \frac{H}{\rho C_p} \tag{7}$$

where u is the velocity. You can understand why the advection term depends on the *gradient* of T if you think about contour lines (e.g. of temperature) being carried past you.

Example How much power is carried by the Gulf Stream? If all this power was used to evaporate water, what would be the vapour production rate?

Example What kind of geothermal flow rate ($\mathrm{m}^3\ \mathrm{s}^{-1}$) would be required to replace a standard (500 MW electric) fossil-fuel power station?

Balancing advection against conduction in equation (7) we obtain

$$\frac{uT}{d} \sim \frac{\kappa T}{d^2}$$

which gives us a dimensionless number ud/κ (known as the Peclet number). This tells us whether advection or conduction is more important.

Example How deep beneath the surface is an undisturbed geotherm reached if the surface is eroding at 10 mm/a?

Example How fast would magma have to ascend from 10 km depth up a dike 1 m wide if it is not to freeze before reaching the surface?

Example Are there any circumstances under which conductive heat transport is important in the Earth's atmosphere?

Note: Although a naive examination of the Peclet number for e.g. the Earth's mantle would suggest that conduction is completely unimportant compared with advection, in fact it cannot be neglected. This is because *boundary layers* develop, across which the heat flow is conductive (velocities are small). We will deal with **convection**, and the development of convective boundary layers, in a subsequent lecture.

Balancing advection against heat production in equation (7) we obtain

$$T \sim \frac{Hd}{\rho C_p u}$$

Example Back to our elephant: what is the flow rate of fluid (blood) transporting heat? You'll have to correct for the small area occupied by blood vessels.

Radiation

Energy can be carried by photons. The black body heat flux emitted by a body at temperature T (in K) is given by

$$F = \epsilon \sigma T^4 \tag{8}$$

where ϵ is the emissivity (≈ 1) and σ is Stefan's constant. The wavelength at which the maximum power occurs goes as $1/T$. The fraction of photon energy absorbed is given by $1 - A$, where A is the *albedo* (1=perfectly reflective).

Example Use the surface temperature of the Earth to estimate the power output of the Sun. What is the Sun's surface temperature?

Example What surface area would need to be covered with solar cells to satisfy the energy demands of the USA?

We can balance advection against radiation in equation (7) to obtain a dimensionless number

$$\frac{\epsilon \sigma T^3}{\rho u C_p}$$

Example Do forest fires transmit more heat by advection or by radiation?

Useful numbers

$\kappa \approx 10^{-6}$ m^2 s^{-1} (rock, ice) 10^{-5} m^2 s^{-1} (metals, gases)
$\rho \mathrm{C_p} \approx 3$ MJ m^{-3} K^{-1} (solids) 3 ρR (gases, per mole)
$\mathrm{L_{vap}} \approx 1$ MJ/kg, $L_{fus} \approx 0.1\ MJ/kg$
$\sigma \approx 6 \times 10^{-8}$ W m^{-2} T^{-4}