Turbulence

Governing Equations

The simplified Navier-Stokes equation (representing three equations, one for each spatial dimension) can be written as:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g} \tag{1}$$

where velocity $\mathbf{u}$ and gravity $\mathbf{g}$ are vectors, and pressure p is a scalar (but recall that the gradient of a scalar, like pressure or elevation, is a vector). Physically, these equations represent conservation of momentum (or alternately a way of re-writing Newton's Second Law). On the LHS, the first term represents accumulation of momentum in a control volume, while the second term represents the net advection of momentum into the control volume. On the RHS, the first term is the pressure gradient force, the second is the dissipation of momentum by fluid viscosity, and the last is the familiar gravitational force. The term that causes these equations to be non-linear is the advection term $\mathbf{u} \cdot \nabla \mathbf{u}$. If we non-dimensionalize Eq. 1 using a characteristic velocity and length scale, it can be re-written as:

$$\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} = -\nabla \tilde{p} + \frac{1}{\mathrm{Re}} \nabla^2 \tilde{\mathbf{u}} + \mathrm{Fr}^2 \tag{2}$$

where $\tilde{\mathbf{u}}$ is non-dimensionalized velocity $\tilde{\mathbf{u}} = \mathbf{u}/u_c$ where u_c is the characteristic velocity, and p has been normalized by ρu_c^2. In this form, one can see the controlling effect of Re and Fr on the flow. If Re is very large, then advection dominates viscous dissipation, which is the turbulent regime. If Re is very small, then viscosity dominates advection, and the non-linear term can be neglected, leaving a much more tractable linear problem that can be solved exactly in many cases. If gravity cannot be neglected, then Fr will play a significant role as well. Later in the notes, we will break the velocity vector into its x, y, and z component velocities, which we will denote as u, v, and w, respectively.

Also note that Eq. 1 is very similar to the heat transfer equation given previously, but instead of temperature T as the dependent variable, we have velocity u (Eq. 1 also has pressure and gravity terms that do not have direct analogs in heat transfer). All the order-of-magnitude estimations that were applicable for heat transfer can also be used here.

Scaling Relationships

There isn't a good definition of turbulence, but what seems to be generally agreed upon is that turbulence is a *chaotic flow*. *Chaotic* implies sensitivity to initial conditions, but does not imply purely stochastic (non-deterministic); indeed it is argued that turbulence is deterministic (i.e. the current state permits prediction of a future state) because the Navier-Stokes equations are deterministic, although the time scale for prediction may be extremely short. In addition, turbulence is a property of a flow, and not a fluid. The classical picture of turbulence can be summarized as:

- Kinetic energy is input to a system at large length scales (comparable to that of the system).

- This drives eddies with length scales comparable to that of the system.
- Energy is passed to smaller length scales by breaking up larger eddies into smaller eddies (the energy cascade).
- Eventually, the energy is passed to a length scale where viscosity is important, and the kinetic energy of eddies at this scale is converted to heat by viscous dissipation.

In classical turbulence, energy can *not* be transferred from smaller scales to larger scales, only vice versa. We'll define some quantities of interest:

ε: energy dissipation rate. It is the kinetic energy dissipated by viscosity per unit time per unit mass of fluid; it therefore has units of $\mathrm{W\,kg^{-1}}$, which can be rewritten as $\mathrm{m^2\,s^{-3}}$. According to classical turbulence, then, this is the rate at which energy is passed down from large scales to small scales (where it is dissipated) via the energy cascade and **is invariant with length scale ℓ.**

ℓ: length scale. Ranges from L, the **overall length scale** of the system, to η, the **Kolmogorov (dissipation) length scale**. In between these scales is the **inertial subrange**, where $\ell \ll L$ and also $\ell \gg \eta$.

k: wave number, which is simply the inverse of length scale, i.e. $k \sim \ell^{-1}$ (officially it's $k = {}^{2\pi}/_{\ell}$).

u_ℓ: velocity scale associated with length scale ℓ. More specifically, this is "the r.m.s. value of the velocity subject to bandpass filtering, say of an octave around the wavenumber ℓ^{-1}" (Frisch, 1995).

t_ℓ: time scale associated with scale ℓ, often termed the 'eddy turnover time', and can be defined as $t_\ell \sim \ell/u_\ell$. This is the typical time scale for an eddy of length scale ℓ to undergo "significant distortion" (Frisch, 1995). This is also the typical time scale for the transfer of energy from scale ℓ to smaller scales, since this distortion is the mechanism for energy transfer.

E_ℓ: turbulent kinetic energy (TKE) associated with length scale ℓ.

$Re(\ell)$: Reynolds number associated with length scale ℓ.

ν: kinematic viscosity, units of diffusivity, i.e. $\mathrm{m^2 s^{-1}}$.

We'll use simple dimensional analysis to determine the relationships among these different quantities. TKE is defined as:

$$E_\ell \sim u_\ell^2 \tag{3}$$

The energy dissipation rate ε can be written as a combination E_ℓ and t_ℓ, or of u_ℓ and ℓ:

$$\varepsilon \sim \frac{E_\ell}{t_\ell} = \frac{u_\ell^3}{\ell} \tag{4}$$

Eqs. (3) and (4) are true across all length scales. If we re-arrange Eq. (4) for velocity, remembering that ε is constant over ℓ, we find

$$u_\ell \sim (\varepsilon \ell)^{1/3} \tag{5}$$

which tells us that velocity, and hence TKE, *increases* with length scale. The eddy turnover time is simply:

$$t_\ell \sim \frac{\ell}{u_\ell} \tag{6}$$

At the Kolmogorov (or dissipation) length scale η, viscosity converts TKE to heat. Thus, η must depend on ν and ε, which from dimensional analysis and some physical reasoning, leads us to deduce the relationship:

$$\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \tag{7}$$

The Reynolds number at any length scale is defined just as described previously:

$$Re(\ell) = \frac{u_\ell \ell}{\nu} \tag{8}$$

For many geophysical flows (atmospheres and oceans), $Re(L)$ is very large. For example, for the boundary layer of Earth's atmosphere, we have typical values of $u_L = 1\,\mathrm{m\,s^{-1}}$, $L = 10^3\,\mathrm{m}$ and $\nu = 10^{-5}\,\mathrm{m^2\,s^{-1}}$, which yields $Re(L) = 10^8$. The Kolmogorov scale is defined as the length scale where viscosity and kinetic energy are comparable, which means that, $Re(\eta) \sim 1$, which is consistent with Eq. (7). Combining Eqs. (7) and (8) also yields:

$$Re(\ell) \sim \left(\frac{\ell}{\eta}\right)^{4/3} \tag{9}$$

which gives an alternate interpretation of $Re(\ell)$ as related to the ratio of length scales.

Examples: characteristic turnover time in ocean mixed layer; the path of a butterfly; limit of physical erosion in rivers; decay of turbulence in a box; energy dissipation in a cloud; extent downwind that turbulence from a mountain range is felt; how fast does a cook have to whisk a vinaigrette

Energy Cascade

One of the predictions of Kolmogorov 1941 theory is the gradient in TKE from large to small scales. Just like heat flows from high to low temperature, kinetic energy "flows" from the source (eddies with length scale L) to the sink (eddies with scale η). In between, i.e. in the inertial subrange, there must a gradient in TKE to cause this flow, i.e. $dE/d\ell$, or equivalently, $E = E(\ell)$. Combining Eqs. (3) and (5) yields:

$$E(\ell) \sim \varepsilon^{2/3} \ell^{2/3} \tag{10}$$

which is a famous prediction of turbulence theory. Fig. 1 shows measurements of $S_2(\ell)$, which is equivalent to $E(\ell)$, versus ℓ, and demonstrates that this theory works very well.

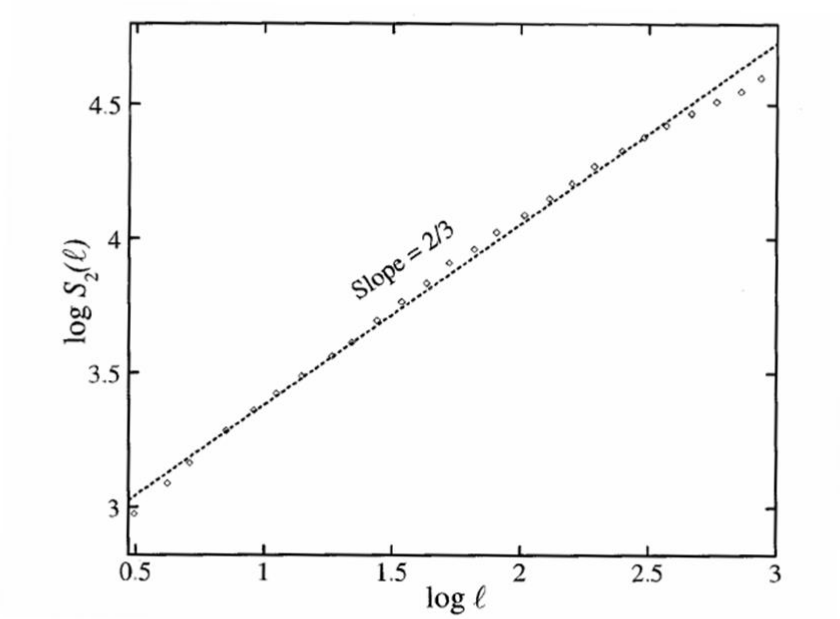

Figure 1: 2/3 Law

Sometimes, Eq. (10) is transformed by two simple operations. First, ℓ is replace with its inverse, wavenumber k. This yields:

$$E(k) \sim \varepsilon^{2/3} k^{-2/3}$$

Second, $E(k)$ is re-written as an energy *density* $e(k)$, i.e. energy per unit wavenumber (also known as the *power spectrum*), such that:

$$e(k) = \frac{E(k)}{k} \sim \varepsilon^{2/3} k^{-5/3} \tag{11}$$

which yields the famous "5/3 power law" dependence of $e(k)$on k (Fig. 2).

Reynolds-Averaged Equations and Turbulence Closure

Because turbulence is a chaotic system, we are not usually interested in predicting the details of the flow field; rather we are often interested in predicting the *statistical mean* of the flow fields, such as mean velocities. To understand how this is usually done, let's write an equation similar to Eq. (1) but for energy rather than momentum (we do this because it's easier to visualize temperature fields than momentum vector fields), which yields:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S \tag{12}$$

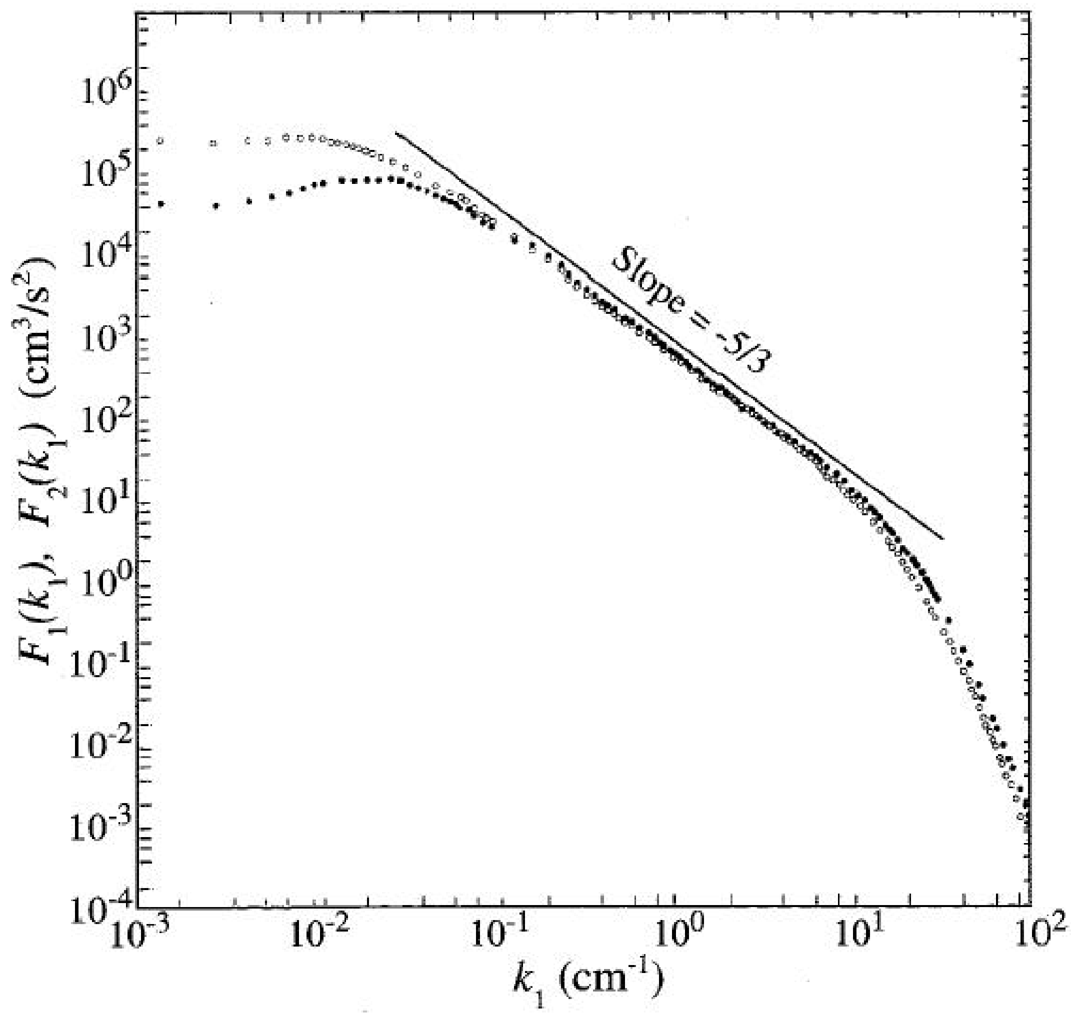

Fig. 5.7. log–log plot of the energy spectra of the streamwise component (white circles) and lateral component (black circles) of the velocity fluctuations in the time domain in a jet with $R_\lambda = 626$ (Champagne 1978).

Figure 2: 5/3 Law

where S are sources of energy (e.g. radiation). Now, we decompose the temperature $T = T(x, y, z, t)$ into two components (this process is called *Reynolds decomposition*):

$$T(x, y, z, t) = \overline{T}(x, y, z) + T'(x, y, z, t) \tag{13}$$

where $\overline{T}$ is the time-average of temperature (sometimes called the steady component of T) and T' is fluctuating component of T around $\overline{T}$. We do this because we're mainly interested in predicting the mean temperature in a flow $\overline{T}$, and would prefer to ignore the random fluctuations caused by turbulence as represented by T'. Important properties of $\overline{T}$ and T' are:

$$\frac{d\overline{T}}{dt} = 0 \text{ and } \overline{T'} = 0 \tag{14}$$

Now, if we use the Reynolds decomposition from Eq. (13) and substitute into Eq. (12), and do the same for all the velocity components (u, v, w), and see what we get on the LHS::

$$\text{LHS} = \frac{\partial\left(\overline{T} + T'\right)}{\partial t} + \left(\overline{u} + u'\right)\frac{\partial\left(\overline{T} + T'\right)}{\partial x} + \left(\overline{v} + v'\right)\frac{\partial\left(\overline{T} + T'\right)}{\partial y} + \left(\overline{w} + w'\right)\frac{\partial\left(\overline{T} + T'\right)}{\partial z}$$

Now we take the time-average of the entire equation, since that's what we're interested in, again look only at the LHS::

$$\overline{\text{LHS}} = \overline{\frac{\partial\left(\overline{T} + T'\right)}{\partial t} + \left(\overline{u} + u'\right)\frac{\partial\left(\overline{T} + T'\right)}{\partial x} + \left(\overline{v} + v'\right)\frac{\partial\left(\overline{T} + T'\right)}{\partial y} + \left(\overline{w} + w'\right)\frac{\partial\left(\overline{T} + T'\right)}{\partial z}} \tag{15}$$

$$= 0 + \left(\overline{u}\frac{\partial\overline{T}}{\partial x} + \overline{v}\frac{\partial\overline{T}}{\partial y} + \overline{w}\frac{\partial\overline{T}}{\partial z}\right) + \left(\frac{\partial}{\partial x}\overline{u'T'} + \frac{\partial}{\partial y}\overline{v'T'} + \frac{\partial}{\partial z}\overline{w'T'}\right) \tag{16}$$

where a lot of simplification has happened between steps 1 and 2, as explained below. The two terms in parentheses have different physical interpretations. The first represents the **mean advection of energy by the mean wind**, and is presumably what we're interested in. The second term, however, is the **net turbulent transport of energy, and is non-zero!** We'll take a brief aside to show how we get from step 1 to 2 in deriving Eq. (16).

Turbulent Transport

Let's look at the consequences of Reynolds averaging on a term that is simply the product of velocity and temperature, uT:

$$\overline{uT} = \overline{(\overline{u} + u')(\overline{T} + T')} = \overline{(\overline{u}\overline{T} + \overline{u}T' + u'\overline{T} + u'T')} = \overline{u}\overline{T} + \overline{u'T'} \tag{17}$$

where $\overline{\overline{u}T'} = \overline{u'\overline{T}} = 0$ because $\overline{u'} = \overline{T'} = 0$. The $\overline{u'T'}$ term is *not* zero, however, if fluctuations in u and T are *correlated*, which is almost always the case. This is easily imagined in the case of convection of water in a pot on the stove, where u is the vertical velocity, and T is the temperature. A small parcel of water that has a $u > \overline{u}$ (i.e. $u' > 0$) will, because the convection is buoyancy driven, have a tendency to have a higher T as well, i.e. $T' > 0$; conversely $u' < 0$ has a correlation with $T' < 0$. Therefore, the product $u'T'$ in both cases is greater than zero, and thus its time average $\overline{u'T'}$ will be non-zero and positive! Thus, $\overline{u'T'}$ represents the *turbulent transport* of temperature T.

In general, turbulent transport of any quantity a is not-negligible where the turbulence occurs in a flow where there is a spatial gradient of a.

Examples: temperature gradient in a floor-heated home; what is the vertical gradient in CO2 in the boundary layer; what is the minimum surface nutrient uptake by phytoplankton (what does this correspond to in carbon export?);

Turbulence Closure

Reynolds averaging leads to a problem in solving the Navier-Stokes or other non-linear equations: the transport of, say, sensible heat energy depends not only on the mean flow (i.e. $\overline{u}$), but it also depends on turbulent quantities like $\overline{u'T'}$. This means that in while we started with *two* unknowns in our toy problem of Eq. (17), u and T, we now have *three* that we must solve for: $\overline{u}$, $\overline{T}$, and $\overline{u'T'}$. But the number of equations remains two: conservation of mass and energy. Therefore, any set of Reynolds-averaged equations is not closed - the fundamental physics isn't sufficient to define the problem. To close the problem, we need one additional equation. The easiest way to do this is to somehow write $\overline{u'T'}$ in terms of $\overline{u}$ and $\overline{T}$ so we can eliminate this term from the equations and solve for what we (probably) most care about, $\overline{u}$ and $\overline{T}$. However, there is no fundamental law (i.e. based on conservation of mass, momentum or energy) that tells us how to do this (or, for that matter, that tell us this is even possible); therefore any attempts to do this are semi-empirical at best and are thus *parameterizations* of turbulent flow.

Horizontal and Vertical Turbulent Transport Simplifications

Let's return to the larger problem of Eq. (16). We can simplify the turbulent transport term using dimensional analysis. We start by considering a boundary layer of depth H over a surface whose properties vary in the horizontal on a length scales L_s. For the atmosphere, H is typically O(1 km), while in many places, L_s can be O(10^1 to 10^3 km) so this is often a good approximation. This implies that the largest eddies have a characteristic size H. Lastly, if the turbulent fluctuations in velocity and temperature have a characteristic value u_f (which in the eddy model of turbulence must be the same in the horizontal and vertical!) and T_f, then we can re-write the turbulent transport term from Eq. (16) as:

$$\frac{\partial}{\partial x}\overline{u'T'} + \frac{\partial}{\partial y}\overline{v'T'} + \frac{\partial}{\partial z}\overline{w'T'} \sim \frac{u_f T_f}{L_s} + \frac{u_f T_f}{L_s} + \frac{u_f T_f}{H} \sim \frac{u_f T_f}{H}$$

where the first two terms have been dropped because $H \ll L_s$. This shows that vertical turbulent transport is much more important than the horizontal components in the energy budget, and we therefore can use this result to simplify the original equation Eq. (12) to:

$$\underbrace{\frac{\partial \overline{T}}{\partial t}}_{\text{allowed?}} + \overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} + \overline{w}\frac{\partial \overline{T}}{\partial z} + \frac{\partial}{\partial z}\overline{w'T'} = \kappa\left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2} + \frac{\partial^2 \overline{T}}{\partial z^2}\right) + S$$

where we've now done something semi-evil and re-introduced a term that we had initially zeroed out, i.e. $\frac{\partial \overline{T}}{\partial t}$. You might ask: if $\overline{T} = \overline{T}(x, y, z)$, then it should logically follow that $\frac{\partial \overline{T}}{\partial t} = 0$ as we already claimed in Eq. (14). The answer is hand-waving (and some people think it shouldn't be allowed, period) but we will find it useful for OOM estimation: as long as we're interested in long-term changes in $\overline{T}$, then $\frac{\partial \overline{T}}{\partial t}$is small and we haven't introduced any horrible contradictions.

If we now align the horizontal x-axis with the mean wind (**changing the problem to 2-D!**), and assume that $\overline{w} = 0$ (is this reasonable?), and further assume that diffusion is slow compared to advection and turbulence (can you show this?), we finally arrive at:

$$\frac{\partial \overline{T}}{\partial t} + \overline{u}\frac{\partial \overline{T}}{\partial x} + \frac{\partial}{\partial z}\overline{w'T'} = S \tag{18}$$

There are times when one has to retain the diffusion term, which you can determine by OOM analysis.

Mixing Length Theory

One way to close the equations is by using mixing length theory, which is the most common parameterization of turbulent mixing currently used in large-scale numerical models such as GCMs. We idealize eddies as taking random fluid parcels from some level, and advecting them up or down over some characteristic height or mixing length h at some characteristic speed w_f, where the fluid parcel gets homogenized with the other air at that level. We assume that turbulent transport is caused primarily by eddies whose length scale $h \sim L$ (since the majority of the TKE is at large length scales) and therefore v is a characteristic large-eddy velocity. At any height z, half the time there is an updraft with $w_{up}^{'}$ carrying fluid upward from an average height $z - {}^{h}\!/_{2}$, and the other half of the time there is a downdraft with $w_{dn}^{'}$ carrying fluid downward from an average height $z + {}^{h}\!/_{2}$, and we assume that $w_{up}^{'} = -w_{dn}^{'} = w'$. Consider the corresponding vertical flux of some advected quantity a (such as energy, momentum or Francis' aftershave). In updrafts:

$$a_{up}^{'} = \overline{a}\left(z - {}^{h}\!/_{2}\right) - \overline{a}(z)$$

If we assume that a varies roughly linearly between $z - {}^{h}\!/_{2}$and z, then:

$$a_{up}^{'} \simeq -\frac{h}{2} \cdot \frac{d\overline{a}}{dz}$$

Similarly in downdrafts:

$$a_{dn}^{'} = \overline{a}\left(z + {}^{h}\!/_{2}\right) - \overline{a}(z) \simeq \frac{h}{2} \cdot \frac{d\overline{a}}{dz}$$

Thus, the Reynolds average is:

$$\overline{w'a'} = \frac{1}{2}\left(w_{up}^{'}a_{up}^{'} + w_{dn}^{'}a_{dn}^{'}\right) = \frac{1}{2}\left[w'\left(-\frac{h}{2} \cdot \frac{d\overline{a}}{dz}\right) + (-w')\,\frac{h}{2} \cdot \frac{d\overline{a}}{dz}\right] = -\frac{w'h}{2} \cdot \frac{d\overline{a}}{dz}$$

which can be re-written as:

$$\overline{w'a'} = -K_a \frac{d\overline{a}}{dz} \tag{19}$$

where $K_a = {}^1\!/\!_2\, w'\, h$ is defined as the **eddy diffusivity** of property a. Note that it has the same units as diffusivity, but is related to a turbulent velocity w' and a length scale h, and that the overall form of the equation is exactly the same as that of Fick's Law of diffusion. For the atmospheric boundary layer, typical values for w' and h are 1 m/s and 1 km, which predicts $K_a = 500\text{m}^2\text{s}^{-1}$, which is much larger than molecular diffusion! Notice that we have now closed the turbulence problem by re-writing the turbulent fluctuation term $\overline{w'a'}$ in terms of mean properties (well, sort of - there's the niggling issue that K_a depends on a characteristic eddy velocity w'; there are formulations for K_a in certain classes of problems that eliminate this issue, such as in boundary layers as we'll discuss later).

Examples: Francis' aftershave revisited; latent heat transport in boundary layer; how far downstream of a river bottom toxic waste dump is the surface water contaminated; time scale for energy to be transported through the Sun's convective zone; spreading of a smoke plume.

Ekman Spirals

One good application of mixing length theory is Ekman spirals. If we write the equivalent to Eq. (18) but for momentum in the x-direction, we get:

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \frac{\partial}{\partial z}\overline{w'u'} = S \tag{20}$$

where now S represents sources of momentum change. On the LHS, if we assume a steady flow, $\frac{\partial \overline{u}}{\partial t}$ is zero. If we assume that at any height, $\overline{u}$ is constant, then $\frac{\partial \overline{u}}{\partial x}$ goes to zero. That leaves us, now, with:

$$\frac{\partial}{\partial z}\overline{w'u'} = S = \frac{1}{\rho}\frac{dP}{dx} + f_c\overline{v} = f_c(\overline{v_g} - \overline{v}) \tag{21}$$

where the two source terms represent a pressure gradient and a coriolis force, and f_c is the coriolis parameter

$$f_c = 2\,\Omega\,\sin\,\lambda$$

where Ω is the Earth's rotation rate and $\lambda$is the latitude; f_c has a typical value of 10^{-4} s^{-1}. We have also defined in Eq. (21) the *geostrophic wind* $\overline{v_g}$ as:

$$\overline{v_g} = \frac{1}{f_c\rho}\frac{dP}{dx} \tag{22}$$

If we now use the equivalent of Eq. (19) and substitute into Eq. (21), we get:

$$K_a\frac{d\overline{u}}{dz} = f_c(\overline{v_g} - \overline{v})$$

Physically, what this equation is telling us is that turbulent transport of **x-direction** momentum leads to changes in the coriolis force, which then causes a change in the **y-direction** momentum. So once we have the right equation, we can estimate the depth of the Ekman layer using simple dimensional analysis, using the knowledge that the only relevant parameters are K_a and f_c:

$$h = \sqrt{\frac{K_a}{f_c}}$$

Substituting for the atmosphere yields an estimate of $h = 2$ km, which is a very nice estimate of the *planetary boundary layer thickness*! For the ocean, the depth of the *mixed layer* is smaller because the vertical turbulent transport is lower (how much lower?).

Realistic Vertical Transport

The types of turbulence that we've been discussing are situations where the fluid is neutrally stable (in the vertical, obviously), i.e. follows an adiabat. In such a fluid, if a temporary burst of energy displaces a parcel of fluid upwards by some distance, then its final resting place will be the location when the burst of energy is over (or maybe more specifically, after all the kinetic energy from that burst has dissipated). However, in a stable fluid, that parcel would return to its initial vertical location after it loses all of its kinetic energy. More importantly, in an unstable fluid, the initial motion will create a much larger displacement than one would predict from the size of the initial burst alone. This latter case is a more realistic view of, say, vertical transport in the atmosphere where *convection* dominates, rather than pure shear-driven turbulent transport. As we might deduce from watching hang gliders, localized but strong *thermals* are responsible for the majority of vertical transport of energy, water, etc., in the atmosphere. In the ocean, primary downwelling regions such as in the Southern Ocean and the North Atlantic dominate vertical transport. This is a whole separate topic that we won't address right now!