ES 265 Order of Magnitude Phys & Chem
Lecture Outline 5
Oscillators and Waves

Oscillators and waves crop up in all sorts of contexts, from pendula and ocean waves to electromagnetic waves and planetary librations. Generally, an oscillator requires an equilibrium state and a restoring force if it deviates from that state. Another useful way of thinking about oscillations is that they involve a periodic conversion of one kind of energy (often kinetic) to another (often potential), and back again.

The most useful characteristic of an oscillator is its period T, or more usually its *angular frequency*, $\omega = 2\pi/T$.

Part of the reason for focusing on period rather than velocity is that some (*dispersive*) waves have two kinds of velocities, group and phase velocities, which may differ from each other.

If a physical oscillator is displaced from its equilibrium state, it will experience a restoring force. For *simple harmonic motion*, the restoring force is proportional to the displacement, that is $F = -kx$ where k is the *spring constant*. This force generates an acceleration $F/m = \frac{d^2x}{dt^2}$. The result is an oscillation with a frequency independent of the displacement:

$$\omega^2 = \frac{k}{m} \tag{1}$$

The frequency depends on the spring constant (restoring force per unit displacement) and the mass (resistance to that force). Most of the difficulty in oscillators lies in determining the correct equivalents to k and m. For instance, in *torsional* systems, we need to consider the torque per unit displacement (equivalent of k), and the moment of inertia of the system (equivalent of m).

To derive ω another way, consider the energies involved. The maximum potential energy is $\frac{1}{2}kx^2$. The maximum kinetic energy is $\frac{1}{2}mv^2$. We need to convert from velocity to frequency, which we do by writing $v \sim x\omega$. *Equating potential energy to kinetic energy*, we arrive at equation (1). The energy approach is sometimes easier, but one should generally convert v to ω first.

Pendulum

For small angular displacements θ, the restoring force is $mg\theta \approx mgx/l$ where l is the pendulum length. So the restoring force per unit displacement is mg/l and the mass is m, giving us

$$\omega^2 = \frac{g}{l} \tag{2}$$

In terms of energy, the maximum potential energy is $mgh \approx mgl\theta^2$. The kinetic energy is $mv^2 \approx ml^2\theta^2\omega^2$, making use of the relationship $v \sim x\omega$. Again, putting these two together, we get equation (2).

Example How much faster does a six-foot person cover the ground than a five-foot person?

Example: Planets and satellites wobble backwards and forwards slightly (librations). The tidal torque per unit angular displacement of a satellite is given by $\sim GM_p(C-A)/a^3$ where a is the semi-major axis, G is the gravitational constant, M_p is the primary mass and $(C-A)$ is the moment of inertia difference for the satellite. How does the libration frequency compare with the orbital frequency?

Continuous materials

For continuous materials, it is often useful to replace the spring constant k with EA/l, where E is Young's modulus or some other measure of rigidity having units of pressure, A is cross-sectional area and l is a lengthscale (for waves, this is usually the wavelength).

We have

$$\omega^2 \sim \frac{EA}{lm} \sim \frac{1}{l^2}\frac{E}{\rho} \tag{3}$$

where ρ is the density.

Example What is the lowest frequency of normal mode oscillations for the Earth?

Example Derive an equivalent expression to (3) for a fluid body. What is the lowest frequency of normal mode oscillations for the Sun?

Waves

A wave is a propagating oscillation. For a particular frequency of wave, the propagation velocity of that wave is called the *phase velocity* and is given by

$$v_{ph} = \frac{\omega}{k} = f\lambda \tag{4}$$

where f is the frequency (in Hz), $k = 2\pi/\lambda$ and λ is the wavelength.

If the *wave packet* consists of waves of more than one frequency, then the propagation velocity of the packet of waves, the *group velocity*, is given by

$$v_{gr} = \frac{\partial \omega}{\partial k}$$

If waves are *dispersive*, then the group velocity and the phase velocity are not the same thing.

If we take equation (3) and assume that $l \sim \lambda$, then we can use equation (4) to derive the (phase) velocity for sound waves in an elastic solid:

$$v^2 \sim \frac{E}{\rho}$$

So in theory seismic wave velocities are frequency-independent i.e. non-dispersive. (In practice surface waves *are* dispersive, because different wavelengths sample to different depths, and both E and ρ vary with depth).

Example How does this (macroscopic) definition of sound waves relate to an atomistic description of sound transport?

Example What is the equivalent expression for sound waves in air? Again, how do we relate this answer to an atomistic description?

Water Waves

In the *deep water* limit, $\lambda \ll D$ where D is the ocean depth and the waves do not feel the influence of the seafloor. The column of water responding to the surface disturbance has a thickness $\approx \lambda$.

Moving the water from a flat state to waves with an amplitude h involves a potential energy $\sim \lambda h^2 g\rho$ per unit width.

In treating the kinetic energy, we have to be careful with velocity. The upwards velocity is ωh and extends to a depth λ, so the kinetic energy per unit width is $\sim \lambda^2 \rho \omega^2 h^2$. Equating the two terms we obtain

$$\omega^2 \sim \frac{g}{\lambda}$$

which can also be rearranged using equation (4) to give us $v_{ph}^2 \sim g\lambda$. This latter result shows that deep water waves are *dispersive* - the velocity depends on the wavelength.

Example: A conventional ship sets up a bow wave which is roughly equal to the length of the ship. What is the maximum speed of a small motor boat, and what is the Froude number at this speed? How might this limit be avoided?

In the *shallow water* approximation, where $\lambda \gg D$, the waves do feel the influence of the seafloor. The same results as above hold, except that D replaces λ. Thus $v_{ph}^2 \sim gD$, so that shallow water waves are non-dispersive.

Example: A tsunami with a 5 km wavelength is a shallow-water wave, even in the central Pacific ocean. How fast does it travel? What does conservation of mass imply about the behaviour of this wave as it approaches the shore?

Similar results hold if *surface tension* γ rather than gravity is the dominant restoring force. The extra energy per unit width required to take a piece of water from flat to sinusoidal is $\sim \gamma h^2/\lambda$, by Pythagoras. Equating this potential energy to kinetic energy as before, we obtain for deep water

$$\omega^2 \sim \frac{\gamma}{\lambda^3 \rho}$$

Example What is the period of surface-tension driven ripples in shallow water?

Example What is the characteristic frequency of a bubble expanding and contracting underwater (cavitation)?

Example One can imagine planetary oscillations in which contracting material releases gravitational energy, causing heating and expansion. What determines the period of these oscillations?

Example What is the Brunt-Vaisala frequency of a fluid? This effect occurs when a stably stratified fluid is perturbed from an initial density gradient $d\rho/dz$.