ES111 Mathematical Methods in the Earth Sciences
Lecture Outlines #14
Eigenvectors and Eigenvalues

Eigenanalysis characterizes the spatial geometry of square matrices, affording insight into their effect as transforms, inverses, etc. There is a massive literature built up around this topic; our discussion will be brief and focused on eigenanalysis in strain theory.

Eigenanalysis involves two things: eigenvectors and eigenvalues. What are they? Well, both are defined by a single vector equation:

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where **A** is a square matrix, **x** is an *eigenvector* of **A** and λ is the associated *eigenvalue* (there may be more than one eigenvector per eigenvalue). λ is a scalar, thus this equation tells us that **A** does not alter the direction of its eigenvectors, it only scales their length. Thus the eigenvectors of **A** define its *characteristic directions.*

How do we obtain the eigenvectors and eigenvalues of a matrix? The answer is pretty simple. First we rewrite the above equation as:

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Aha! For there to be a solution of this equation, the matrix in parenthesis must have a null space. If it has a null space, it can't be full rank and its determinant must be zero! Thus

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

provides us with a polynomial in terms of λ that can be solved for the eigenvalues of **A**. If **A** is *n* by *n*, the polynomial has terms up to λ^n and will have from 1 to *n* distinct roots, corresponding to the eigenvalues of **A**.

The eigenvalues of **A** have several unique properties:

1. The sum of elements along the main diagonal of **A** (a quantity known as the trace of **A**) equals the sum of eigenvalues (repeating repeated roots).

$$\text{Trace}\,\mathbf{A} = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

2. The determinant of **A** is equal to the serial product of eigenvalues.

$$\det\mathbf{A} = \prod_{i=1}^{n} \lambda_i$$

   This has an important corrolary. If **A** is singular (not full rank), then it has a zero determinant and must have one or more zero eigenvalues. This means it has a null space since a zero eigenvalue implies

$$\mathbf{Ax} = \mathbf{0}$$

The next question is, of course, how do we obtain eigenvectors? The answer is simple. We plug an eigenvalue into the equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

and solve it for the elements of $\mathbf{x}$. The length of the vector is not determined by the above equation since only $n - 1$ of the n elements of $\mathbf{x}$ are determined. As a result, it is typical to normalized $\mathbf{x}$ to unit length.

If $\mathbf{A}$ is full rank, then the n eigenvectors of $\mathbf{A}$ form a *basis* for n-dimensional space. If $\mathbf{A}$ is not full rank, there are one or more zero eigenvalues and the eigenvectors do not fully span n-dimensional space.

If $\mathbf{A}$ is symmetric and full rank, the eigenvectors of $\mathbf{A}$ form an *orthonormal basis* for n-dimensional space, meaning they span the space and obey

$$\mathbf{x}_i^T \mathbf{x}_j = \delta_{ij}$$

implying that the eigenvectors are orthogonal to one another (note that the eigenvectors are scaled such that they all have unit length).

Matrix Diagonalization:

An n by n full rank matrix $\mathbf{A}$ can be made diagonal via:

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

where $\mathbf{X}$ is composed of the eigenvectors of $\mathbf{A}$, *i.e.*,

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

If $\mathbf{A}$ is symmetric, then $\mathbf{X}$ is a unitary or orthogonal matrix such that

$$\mathbf{X}^{-1} = \mathbf{X}^T$$

hence,

$$\mathbf{D} = \mathbf{X}^T\mathbf{A}\mathbf{X}$$

This last result has an important role to play in strain analysis.

2D Strain:

Assume that the initial position of a point in an undeformed medium is (x,y) and its position after strain has occurred is (x',y'). For small strains, (x',y') can be related to (x,y) by an *Affine Transformation*:

$$ax + by = x'$$

$$cx + dy = y'$$

or $\mathbf{Ax} = \mathbf{x'}$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

This is an example of a linear transformation as discussed in Chapter 7.15 of K. As written, $\mathbf{A}$ is perfectly general. Three canonical forms of the strain matrix $\mathbf{A}$ are:

1. Rigid Rotation

$$\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

2. Simple Shear

$$\mathbf{A} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

Here the shear is parallel to the x axis. γ is referred to as the *shearstrain* and $\gamma = \tan\psi$ defines ψ which is called the *angular shear strain.*

3. Pure Shear

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

a or $d > 1$ implies dilatation (expansion), < 1 implies compression.

The serial product of these three transformations can produce any linear transformation, *i.e.*, any infinitesimal strain field.

Strain Ellipses:

Imagine of a locus of points defining a circle which undergoes some strain,

e_1 and e_2 are the *principlestrains* or *principalextensions*. Given the strain matrix $\mathbf{A}$, we can use matrix diagonalization to obtain the principal strains and the principal *strain axes* PROVIDED there is no simple shear involved. Simple shear does not have a full eigenvector basis. Since pure shear is characterized by a symmetric strain matrix, diagonalizing it is fairly simple. In 2D:

$$\Lambda = \mathbf{S}^T\mathbf{A}\mathbf{S}$$

Here $\mathbf{S}$ contains the eigenvectors of $\mathbf{A}$ normalized to unit length. Λ is a diagonal matrix that has the associated eigenvalues of $\mathbf{A}$ along the diagonal,

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where we have chosen to number the eigenvalues and vectors such that $\lambda_1 \geq \lambda_2$. We can rearrange the above equation, finding

$$\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^T$$

Written this way, the eigenvalues and vectors of $\mathbf{A}$ have simple interpretations in terms of strain. Since $\lambda_1 \geq \lambda_2$,

$$\lambda_1 = 1 + e_1$$

or

$$e_1 = 1 - \lambda_1$$

is the major axis extension and $\mathbf{x}_1$ is the major axis orientation. Likewise

$$\lambda_2 = 1 + e_2$$

or

$$e_2 = 1 - \lambda_2$$

is the minor axis extension and $\mathbf{x}_2$ is the minor axis orientation. Why? Think of strain as evolving by small steps of growth and/or contraction along the major and minor axes. Since these axes are not rotated by $\mathbf{A}$, a circle evolves to an ellipse. In other words, repeatedly applying **A** to a position vector parallel to an eigenvector will not change the direction of the vector, only its length. It will slowly grow or contract depending on the strain matrix.

Note that a general square matrix **A** can be factored into a sum of two matrices, one symmetric, the other skew-symmetric:

$$\mathbf{A} = \mathbf{S} + \mathbf{R}$$

where **S** is symmetric

$$\mathbf{S} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$$

and **R** is skew symmetric

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$$

In strain analysis, we typically deal with **S**, rather than **A**, since it has no rigid rotation in it.