ES111 Mathematical Methods in the Earth Sciences
Lecture Outline #16
First-Order Differential Equations, Part I

We are building a classification scheme for first-order ordinary differential equations. So far, the list contains:

1. *Real Simple:* Hardly a differential equation at all since y doesn't appear.

$$y' = f(x)$$

2. *Separable:* The form of a first-order ordinary separable DE is:

$$g(y)dy = f(x)dx$$

   Since both sides of the equation are total derivatives, we can integrate both to obtain the general solution:

$$\int g(y)\,dy = \int f(x)dx + c$$

3. *Exact:* An ODE is called exact if when written as:

$$M(x,y)\,dx + N(x,y)dy = 0$$

   the condition of exactness is satisfied, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

   where $M(x,y)$ and $N(x,y)$ are functions with continuous first derivatives (so we don't have absolute values, square roots in the denominator, etc.).

We solve exact differential equations by analogy with the total differential. Recall that for some function $u(x,y)$ of x and y, the total differential of u can be written:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

We can cast all exact ODEs as a total differentials by equating the partial derivative of u with respect to x with M, and the partial of u with respect to y with N, *i.e.*,

$$\frac{\partial u}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y} = N(x,y)$$

Which implies that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{is equivalent to} \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

The latter is Clairaut's law and is always true. Thus an exact ODE is the total differential of some function $u(x,y)$. Because the ODE is equated to zero, the total differential of u must also be zero, implying that u is a constant and that $u(x,y) = c$ is the solution of the ODE we seek. We get it by following the same steps we used to recover a function from its total differential. In schematic form, the steps we follow are:

1. Identify M and N and show exactness (show that the partials of M and N are equal).

2. Integrate M with respect to x treating y as a constant. This produces:

$$u(x,y) = \int M dx + f(y)$$

   where $f(y)$ is the integration constant (which can be a function of y since y was treated as a constant during partial differentiation).

3. Compute the partial derivative of u with respect to y and equate it to N to determine $f'(y)$:

$$f'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M dx\right]$$

4. Integrate $f'(x)$ with respect to y. You may ignore the integration constant in this case:

$$f(y) = \int \left[N - \frac{\partial}{\partial y}\left(\int M dx\right)\right] dy$$

5. Lastly, assemble the solution:

$$\int M dx + f(y) = c$$

   which is in implicit form (not solved for y as a function of x or vice versa).

Although I set this up by integrating M with respect to x first, you may also start by integrating N with respect to y and including an integrating factor $g(x)$. Do whichever looks simplest.

Although all real simple and separable equations (in separated form) are exact, not all first-order ordinary equations are. However, all can made exact by multiplication by a function of x, or y, or x and y both. The function is known as an integrating factor. This is because we can treat the resulting equation as a total differential and integrate it as discussed in outline #22.

Integrating factors are not unique, in fact every equation has infinitely many of them. This is good: we only need one, the more there are to choose from, the easier it should be to find one.

The usual approach to obtaining an integrating factor of a non-exact ODE is to assume some general form for it, say $g(x)$ or $f(y)$ and then use the condition of exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

to specify the integrating factor. Note that this approach produces a ODE for the integrating factor which you must solve before solving the original ODE (oh the humanity…). For example, say we start with the first-order ODE:

$$F(x,y)dx + G(x,y)dy = 0$$

where $F(x,y)$ and $G(x,y)$ are continuous functions that don't obey the condition of exactness, *i.e.*,

$$\frac{\partial F}{\partial y} \neq \frac{\partial G}{\partial x}$$

We might try to make the ODE exact by multiplying both sides by a function of y, call it $g(y)$. Thus we seek $g(y)$ such that the equation

$$g(y)F(x,y)dx + g(y)G(x,y)dy = 0$$

is exact, in particular, we want $g(y)$ to satisfy:

$$g'(y)F + g(y)\left[\frac{\partial F}{\partial y} - \frac{\partial G}{\partial x}\right] = 0$$

The hope is that this differential equation is either real simple or, more likely, separable, such that we can solve it for $g(y)$ easily. We then return to

$$g(y)F(x,y)dx + g(y)G(x,y)dy = 0$$

substitute in the (now) known form of $g(y)$ and solve the ODE as an exact equation. It's a lot of work (dealing with integrating factors), but the approach is very general and very powerful. Our next task is to simplify the process of obtaining an integrating factor for a particular subclass of non-exact equations, namely all the linear first-order ODEs.