

ES111 Mathematical Methods in the Earth Sciences

Lecture Outline #4 Lines and Planes

Vectors combined with the dot and cross products provide us with a convenient means of representing lines and planes in 3D.

Lines

Recall that a line has one degree of freedom—specifying position along the line. It is a one-dimensional object in three dimensional space. As such, we should be able to represent it in terms of one free parameter.

Define \mathbf{v} as a vector parallel to the line and let \mathbf{r}_0 be the position vector of any point on the line, then

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

is the *vector equation* for position \mathbf{r} on the line. Here t is the one free parameter—changing its value moves \mathbf{r} up and down the line.

Setting

$$\mathbf{r} = (x, y, z), \mathbf{r}_0 = (x_0, y_0, z_0) \text{ and } \mathbf{v} = [a, b, c]$$

then the vector equation of the line is equivalent to the three scalar equations

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

These are referred to as the *parametric equations*. Notice that there are four unknowns (x , y , z , and t) and three equations, leaving one degree of freedom. Solving for t , yields the *symmetric form*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Planes

Planes have two degrees of freedom since they are two-dimensional objects in 3D. We will need two free parameters to specify position on the plane. We could do this by scaling two vectors and summing them. Instead we will appeal to a single vector—the *normal vector*. Define \mathbf{n} as any vector that is normal to the plane. Being normal to a plane means it is orthogonal to any vector in the plane. Now let \mathbf{r}_0 be the position vector of some point on the plane. The *normal equation* for position \mathbf{r} on the plane is:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

This works since $\mathbf{r} - \mathbf{r}_0$ lies in the plane if \mathbf{r} is on the plane and because \mathbf{n} is orthogonal to any vector in the plane.

If we set

$$\mathbf{n} = [a, b, c], \mathbf{r} = (x, y, z) \text{ and } \mathbf{r}_0 = (x_0, y_0, z_0)$$

Then the normal equation reduces to the *scalar equation*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Notice that there are three unknowns (x , y and z) and one equation, hence two degrees of freedom.

These forms are quite powerful. For example, we can determine the angle of intersection of two planes via the dot product of their normal vectors:

$$\cos = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

We can do the same for lines by using the parallel vectors. We can also easily determine the plane containing two vectors. We obtain the normal vector by crossing the parallel vectors of the two lines. Any point on either line is on the plane, providing the other unknown in the normal equation.