

# ES111 Mathematical Methods in the Earth Sciences

## Lecture Outline #6 Tangent Planes and Total Differentials

### *Tangent Planes*

We can extend the concept of a tangent line to a plane curve (function of 1 variable in 2D) to that of a tangent plane to a space surface (function of 2 variables in 3D). We start with the equation of a plane:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

We solve for  $z - z_0$  and define  $a = -A/C$  and  $b = -B/C$  finding

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

To make this plane tangent to the surface at  $(x_0, y_0)$  we must choose  $a$  and  $b$  correctly. We want the tangent plane to have the same slope as the plane curve in the  $y = y_0$  and  $x = x_0$  planes, thus we choose for  $a$  and  $b$  the partial derivatives of  $f$  with respect to  $x$  and  $y$ , respectively:

$$z - z_0 = \frac{f(x_0, y_0)}{x} (x - x_0) + \frac{f(x_0, y_0)}{y} (y - y_0)$$

Just as we use the tangent line to a plane curve to describe its variation about a point, we can use a tangent plane to describe the variation of a surface about a point. We saw earlier that the tangent line is the first term of a Taylor series expansion of a function. Likewise, the tangent plane is the first two terms of a 2D Taylor series expansion.

### *Total Differential*

For functions of 1 variable, the regular derivative is also a *total differential*. That is for  $y = f(x)$

$$dy = \frac{df}{dx} dx$$

is a total differential. Here  $dy$  describes the variation of the function about a point. Functions of several variables also have total differentials. We get them from the tangent plane.

Let  $z - z_0 = dz$ . Likewise, let  $x - x_0 = dx$  and  $y - y_0 = dy$ . Then for  $z = f(x, y)$

$$dz = \frac{f}{x} dx + \frac{f}{y} dy$$

is the total differential of  $z = f(x, y)$ . It approximates the change in  $z$  for small increments ( $dx, dy$ ) in  $x$  and  $y$ .

We use total differentials for many things. An important application is error analysis.