EART-279
Interpretive Data Processing
Permutations and Combinations

We often need to count the number of simple or composite events that exist in A or the sample space S. For discrete events the two most important counting rules are permutation and combination.

Permutation refers to object ordering. For instance

123 312 231 132 321 213

are the 6 possible permutations of 1, 2 and 3. In general, there are $n!$ permutations of n objects. In our example, one of three numbers can fill the first slot. This leaves two choices for the second slot. When filled, there remains only one token for the third slot, thus the number of permutations is 3 • 2 • 1 = 3! = 6.

When the number of objects used is less than the number in the set, the total number of permutations is n^k for n objects ordered k at a time. This mimics sampling with replacement. The thought process behind this is obvious: think of the objects as digits in a k digit number. The digits are base n. This yields n^k unique sets.

The number of permutations without replacement is obviously smaller. We know that it must equal $n!$ when $n = k$. The rule is:

$$P_k^n = \frac{n!}{(n-k)!}$$

Note the capital P notation: this always denotes permutations of n objects ordered k at a time without replacement.

Permutations are ordered. For instance, {1, 2} is different than {2, 1}. Combinations ignore order; {1, 2} = {2, 1} and the six permutations of 123 are all equivalent.

In general, the number of **combinations** of n objects taken k at a time with replacement is:

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

If we don't allow replacement, we have:

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Combinations are related to binomials through:

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \ \dots \ + \binom{n}{n} p^n = \sum_{i=0}^{n} \binom{n}{i} q^{n-i} p^i$$

Note that this implies:

$$\binom{n}{k} = \binom{n}{n-k}$$

One way to picture this is to note that each combination of k objects drawn from a pool of n leaves behind a combination of n - k objects.

The following table summarizes the counting of n objects sampled k at a time:

Summary

Counting	With Replacement	Without Replacement
Permutations	n^k	$\frac{n!}{(n-k)!}$
Combinations	$\frac{(n+k-1)!}{k!(n-1)!}$	$\frac{n!}{k!(n-k)!}$

Example: How many people does it take in order to have a 50% probability that at least two share the same birthday (day of year only)?

The answer requires coupling two concepts: the binomial distribution and combinations. Let's start with the combinations. Given n people chosen 2 at a time without replacement, we have:

$$c = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

pairs of people. Each pair is a trial with a 1/365 chance of coincident birthdays. The probability of one or more coincident birthdays is governed by the binomials distribution:

$$P(x \geq 1) = \sum_{x=1}^{c} C_x^c \left(\frac{1}{365}\right)^x \left(\frac{364}{365}\right)^{c-x}$$

This is unwieldy. Using the notion of the complement ($x = 0$), we have:

$$P(x \geq 1) = 1 - P(0) = 1 - \left(\frac{364}{365}\right)^c$$

We want this probability to equal or exceed 0.5. Solving for c:

$$0.5 \geq 1 - \left(\frac{364}{365}\right)^c \Rightarrow 0.5 \leq \left(\frac{364}{365}\right)^c$$

$$\ln(0.5) \leq c \ln\left(\frac{364}{365}\right) \Rightarrow c \geq \frac{\ln(0.5)}{\ln\left(\frac{364}{365}\right)} = 252.652$$

Thus the number of pairs must equal or exceed 253. This leads to:

$$253 \le \frac{n(n-1)}{2} \Rightarrow 506 \le n(n-1)$$

And we find that $n = 23$ fills the bill ($c = 506$). Thus any group of 23 people has even odds of at least one repeated birthday. In fact, the odds are much better than this because of holiday sex.